**MATH 579 Exam 9 Solutions**

Part I: A semester is $n$ days long. I will divide these days into one or more chapters of consecutive days, and one day of each chapter I will do a particularly good job teaching. Let $a_n$ represent how many ways there are to do this. Find $a_n$ in closed form.

> For a single chapter of length $n$, there are $n$ ways to pick the day I do a good job. This has generating function $B(x) = \sum_{n\geq 0} nx^n = \frac{x}{(1-x)^2}$. Applying Thm 8.13, we have the desired generating function $A(x) = \frac{1}{1-B(x)} = \frac{1-2x+x^2}{1-3x+x^2} = 1 + \frac{x}{1-3x+x^2} = 1 + \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$, where $\alpha = \frac{3+\sqrt{5}}{2}, \beta = \frac{3-\sqrt{5}}{2}$. We solve the partial fractions to get $A = \frac{1}{\sqrt{5}}, B = \frac{-1}{\sqrt{5}}$. Hence $A(x) = 1 + A\sum_{n\geq 0}\alpha^n x^n + B\sum_{n\geq 0}\beta^n x^n$, so $a_0 = 1$ and $a_n = (\alpha^n - \beta^n)/\sqrt{5}$, for $n \geq 1$.
> NOTE: This turns out to be every other Fibonacci number, but I don't know why.

Part II:

1. Let $a_n$ denote the number of ways to pay $n$ dollars using the usual $1, 5, 10, 20, 50, 100$ dollar bills. Find the generating function for $a_n$.

   By the convolution principle, $A(x) = \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{20})(1-x^{50})(1-x^{100})}$.

2. Find the generating function that can be used to count how many nonnegative integers $a, b, c$ there are with $a+b+c = n$ and (1) $a \leq 2$, (2) $b$ is a multiple of 3, (3) $c$ is odd.

   The g.f. for $a$ is $1+x+x^2$, for $b$ is $\frac{1}{1-x^3}$, for $c$ is $\frac{x}{1-x^2}$ (note: $\frac{1}{1-x^2}$ is for evens, need to shift by 1). Hence by the convolution principle the desired g.f. is $\frac{(1+x+x^2)x}{(1-x^3)(1-x^2)} = \frac{x}{(1-x)^2(1+x)}$.

3. Solve the recurrence $a_0 = 0, a_n = a_{n-1} + 3^n$ using generating functions.

   Multiplying both sides by $x^n$ and summing over $n \geq 1$ we get $A(x) - a_0 = xA(x) + \frac{3x}{1-3x}$, so $A(x)(1-x) = \frac{3x}{1-3x}$ and $A(x) = \frac{3x}{(1-x)(1-3x)} = \frac{-1.5}{1-x} + \frac{1.5}{1-3x}$, using partial fractions. Hence $a_n = -1.5 + (1.5)3^n = (3^{n+1}-3)/2$.

4. Solve the recurrence $a_0 = 2, a_n = 3a_{n-1} - 2$ using generating functions.

   Multiplying both sides by $x^n$ and summing over $n \geq 1$ we get $A(x) - a_0 = 3xA(x) + \frac{-2x}{1-x}$. Hence $A(x)(1-3x) = 2 + \frac{-2x}{1-x}$, $A(x) = \frac{2}{1-3x} + \frac{-2x}{(1-3x)(1-x)} = \frac{2}{1-3x} + \frac{1}{1-x} - \frac{1}{1-3x} = \frac{1}{1-3x} + \frac{1}{1-x}$, using partial fractions. Hence $a_n = 3^n + 1$.

5. What are all the possible ways to number three dice (with positive integers) so that the probability distribution of their total is the same as the probability distribution of three dice with ordinary numbering?

   Each die has generating function $1+x+x^2+x^3+x^4+x^5 = x(1+x)(1+x+x^2)(1-x+x^2)$, and hence their sum has generating function $x^3(1+x)^3(1+x+x^2)^3(1-x+x^2)^3$. We want to divide this into three multiplicands; however each must have a factor of $x$ (else a face would be zero), and each must have a factor of $(1+x)(1+x+x^2)$ (else there would not be six sides, since evaluating at $x=1$ must yield 6). Hence the only issue is how to divvy up the $(1-x+x^2)^3$ part. We can divide it into three equal parts (the usual numbering), we can put all three onto one die (impossible *), or we can have $0-1-2$ as an arrangement. Hence the only alternative numbering comes from $x(1+x)(1+x+x^2), x(1+x)(1+x+x^2)(1-x+x^2), x(1+x)(1+x+x^2)(1-x+x^2)^2$, which is exactly the nonstandard two-dice solution from class (1,2,2,3,3,4)+(1,3,4,5,6,8) and one ordinary die.
   (*)This is actually impossible, since $x(1+x)(1+x+x^2)(1-x+x^2)^3 = x - x^2 + 2x^3 + x^5 + x^6 + 2x^8 - x^9 + x^{10}$, which gives a negative number of sides with a 2 (and 9).

Exam grades: High score=98, Median score=84, Low score=50